Asymptotic dynamics of short waves in nonlinear dispersive models

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Multiple-scale perturbation theory, well known for long waves, is extended to the study of the far-field behavior of *short waves*, commonly called ripples. It is proved that the Benjamin–Bona–Mahony–Peregrine equation can support the propagation of short waves. This result contradicts the Benjamin hypothesis that short waves do not propagate in this model and closes a part of the old controversy over different solutions for the Korteweg–de Vries and Benjamin–Bona–Mahony–Peregrine equations. We have shown that, in a short-wave analysis, a nonlinear (quadratic) Klein-Gordon–type equation replaces the ubiquitous Korteweg–de Vries equation of the long-wave approach. Moreover, the kink solutions of ϕ^4 and sine-Gordon equations are understood as an asymptotic behavior of short waves to all orders. It is proved that the antikink solution of the ϕ^4 model, which was never obtained perturbatively, occurs as a perturbation expansion in the wave number k in the short-wave limit. [S1063-651X(98)05805-X]

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INTRODUCTION

The method of multiple scales, or reductive perturbation method, is a powerful method that allows one to study a large number of physical phenomena, in particular wave motion in nonlinear dispersive systems. It is well known that the far field dynamics of a long wave (LW) with a small amplitude in a nonlinear and dispersive system can almost always be reduced to a small set of model equations such as Boussinesq, Korteweg–de Vries (KdV), modified KdV, etc. [1–4].

The purpose of this work is to look for the far-field behavior of *short waves* (SW) in some nonlinear and dispersive systems. In order to see how SW propagate, we reformulate the method of multiples scales. We will consider three systems. We will first study a system that comes from a hydrodynamic, Benjamin–Bona–Mahony–Peregrine (BBMP) equation. Then, we will consider SW in two important classical relativistic field theory models: The ϕ^4 model (ϕ^4) and the sine-Gordon (SG) equation.

We will prove that, for BBMP (1), SW can build up the same soliton solution as obtained from LW [5]. This raises the question of the unicity of the soliton description. We will prove that the antikink (or kink) solution of the ϕ^4 model (2), which *cannot be obtained* as a perturbative solution in the nonlinearity parameter λ, occurs as a perturbative solution in the wave number k in the SW limit. Moreover, the kink solution of sine-Gordon equation (3) does not enter the classical LW perturbation scheme. We will prove that it appears as a perturbative solution for SW.

SHORT-WAVE APPROACH

Let us consider the problem of the asymptotic dynamics of SW in nonlinear and dispersive systems. All degrees of dispersion of the system are taken into account in a Taylor expansion of the linear dispersion relation $\omega(k)$ around a large value of the wave number k. The asymptotic dynamics of SW for $t\to\infty$ is considered via the introduction of an infinite number of *slow time* variables $\tau_1, \tau_3, \tau_5, \ldots$ and of a *fast space* variable ζ, following the extension theory of Sandri [6]. The use of this fast variable and of an infinite series of slow time variables constitutes the first key of the SW approach.

The solution is expanded in the form of a power series in a small parameter ϵ proportional to the inverse of the wave number k. The perturbative series solution is secular. It is regularized through a renormalization of the frequency. This results from the celebrated Stokes hypothesis on frequency-amplitude dependence in water waves [7].

The Stokes hypothesis is actually the second key tool of our approach. This is explicit here while for LW asymptotic description, the KdV [8,9] or MKdV [10] hierarchies occult the need of this tool, as they naturally provide the correct series expansion of the frequency.

BASIC MODELS

Hence, the problem is the asymptotic behavior of a SW in the Benjamin–Bona–Mahony–Peregrine equation [11] and in two classical relativistic nonlinear models: ϕ^4 and sine-Gordon [12]

$$\text{BBMP:}\quad u_t+u_x-u_{xxt}=3(u^2)_x, \tag{1}$$

$$\phi^4:\quad \phi_{xx}-\phi_{tt}=m^2\phi-\lambda\phi^3, \tag{2}$$

$$\text{SG:}\quad \phi_{xx}-\phi_{tt}=\frac{m^3}{\sqrt{\lambda}}\sin\left[\left(\frac{\sqrt{\lambda}}{m}\right)\phi\right]. \tag{3}$$

The above models have quite different intrinsic characteristics. First SG is an integrable model, whereas BBMP and ϕ^4 are not. Secondly the linear dispersion relation $\omega(k)$ has a finite limit as $k\to\infty$ (SW limit) for BBMP, whereas it is unbounded for SG and ϕ^4.
Indeed we have

$$\omega_{(\text{BBMP})}=\frac{k}{1+k^2}, \tag{4}$$

$$\omega_{(\phi^4)}=\omega_{(\text{SG})}=(m^2+k^2)^{1/2}. \tag{5}$$

The phase and group velocities are all bounded in the SW limit $k\to\infty$. This is the crucial point in our approach, as this very property allows the three models to sustain short waves. Then we face the problem of the *nonlinear propagation of a SW*, which is the object of this work.

THE BBMP MODEL

Let us consider a SW in Eq. (1) characterized by $k=k_0\epsilon^{-1}$ with $k_0\sim\mathcal{O}(1)$ and $\epsilon\ll 1$. The plane-wave solution of the linear problem $u=\exp i\{kx-\omega(k)t\}$ inspires a fast variable $\zeta=\epsilon^{-1}x$ and an infinity of slow time variables $\tau_{2n+1}=\epsilon^{2n+1}t$ $(n=0,1,2,\ldots)$, by expanding ω in powers of ϵ.

We assume the expansion

$$u=u_0+\epsilon^2u_2+\epsilon^4u_4+\cdots \tag{6}$$

and suppose the *extension* $u_{2n}=u_{2n}(\zeta,\tau_1,\tau_3,\ldots)$, $n=0,1,\ldots$, [4,6]. Then, the operators

$$\frac{\partial}{\partial x}=\frac{1}{\epsilon}\frac{\partial}{\partial\zeta}, \tag{7}$$

$$\frac{\partial}{\partial t}=\epsilon\frac{\partial}{\partial\tau_1}+\epsilon^3\frac{\partial}{\partial\tau_3}+\epsilon^5\frac{\partial}{\partial\tau_5}+\cdots \tag{8}$$

allow us to study the behavior of a *short wave* for *large time*.

BBMP gives at orders $\epsilon^{-1},\epsilon,\epsilon^3,\ \ldots$, the equations (written only up to ϵ^3)

$$-u_{0,\zeta\tau_1}+u_0-3u_0^2=0, \tag{9}$$

$$\hat{L}u_2=u_{0,\tau_1}+u_{0,\zeta\zeta\tau_3}, \tag{10}$$

$$\hat{L}u_4=u_{2,\tau_1}-u_{0,\tau_3}+u_{2,\zeta\zeta\tau_3}+u_{0,\zeta\zeta\tau_5}+3(u_2^2)_\zeta, \tag{11}$$

where $\hat{L}$ is the linear operator associated with Eq. (9):

$$\hat{L}(v)=-v_{\zeta\zeta\tau_1}+v_\zeta-6(vu_0)_\zeta. \tag{12}$$

The *unique* solution of Eq. (9) in the form $u_0(\eta)$ with $\eta=k_0\zeta-\omega_1\tau_1-\omega_3\tau_3-\omega_5\tau_5\ldots$, going to zero for $|\zeta|\to\infty$, is

$$u_0=\frac{1}{2}\,\mathrm{sech}^2\,\eta,\quad \omega_1=-\frac{1}{4k_0}. \tag{13}$$

The values $\omega_3,\omega_5,\ldots$, which are the corrections to the principal frequency ω_1 (Stokes' hypothesis) are still free, but will be determined later by the nonsecularity requirement.

Equation (10) for u_2 then reads

$$\hat{L}u_2=\{4\omega_3k_0^2-\omega_1-12\omega_3k_0^2\mathrm{sech}^2\,\eta\}\mathrm{sech}^2\,\eta\,\tanh\,\eta, \tag{14}$$

and its two first right-hand side terms are resonant (secular producing terms) because [13]

$$\hat{L}(\mathrm{sech}^2\,\eta\,\tanh\,\eta)=0.$$

These secular terms are eliminated by choosing

$$\omega_3=\frac{\omega_1}{4k_0^2}=-\frac{1}{4^2k_0^3}.$$

Hence, Eq. (14) yields the solution $u_2(\eta)=4^{-1}k_0^{-2}u_0(\eta)$.

Equation (11) for $u_4(\eta)$ contains secular producing terms originated by the first four terms in the right-hand side. They can be eliminated by choosing $\omega_5=-4^{-3}k_0^{-5}$. The solution is $u_4(\eta)=4^{-2}k_0^{-4}u_0(\eta)$. This procedure can be repeated at any higher order $n=0,1,2,\ldots$ and we obtain recursively

$$u_{2n}(\eta)=\frac{u_0(\eta)}{4^nk_0^{2n}},\quad \omega_{2n+1}=-\frac{1}{4^{n+1}k_0^{2n+1}}. \tag{15}$$

Next, the perturbative series solution (6) can be summed to give

$$u(\eta)=u_0(\eta)\sum_{n=0}^{\infty}\frac{\epsilon^{2n}}{4^nk_0^{2n}}=\frac{4k^2}{4k^2-1}u_0(\eta), \tag{16}$$

and, by using ω_{2n+1}, the argument η in the laboratory coordinates yields

$$\eta=kx+\frac{1}{4k}\sum_{n=0}^{\infty}\frac{t}{(4k^2)^n}=kx+\frac{kt}{4k^2-1}. \tag{17}$$

Therefore, this SW perturbation technique finally leads to the following solution:

$$u(x,t)=-\frac{2k^2}{1-4k^2}\,\mathrm{sech}^2\left[k\left(x+\frac{t}{4k^2-1}\right)\right]. \tag{18}$$

This very expression—solution of BBMP—was obtained in [5] as an asymptotic limit of a LW of small amplitude. Thus, for $t\to\infty$, the nonlinear dynamics of a SW (with an order one amplitude) and of a LW (with small amplitude) are indistinguishable in BBMP. The equation (9)

$$u_{0,\zeta\tau_1}=u_0-3u_0^2$$

is a nonlinear Klein-Gordon equation that substitutes the classical Korteweg–de Vries of the LW approach in this SW approach.

THE ϕ^4 MODEL

The topological antikink-type solution of ϕ^4 will be obtained by perturbation expansion starting from the constant solutions $\phi_0=\pm m/\sqrt{\lambda}$. Hence we seek a solution $\phi(\eta)$ such that $\phi\to\mp m/\sqrt{\lambda}$ for $\eta\to\pm\infty$.

For $\eta<0$, the function $u=\phi-m/\sqrt{\lambda}$ goes to zero for $\eta\to-\infty$ and satisfies

$$u_{xx}-u_{tt}=-2m^2u-3m\sqrt{\lambda}u^2-\lambda u^3. \tag{19}$$

For unidirectional propagation, the convenient fast variable ζ and the slow variables τ_{2n+1} are in this case: $\zeta=\epsilon^{-1}(x-t)$, $\tau_1=\epsilon t,\tau_3=\epsilon^3t,\ \ldots$. Expanding u according to $u=\epsilon^2(u_0+\epsilon^2u_2+\epsilon^4u_4+\cdots)$, the resulting equations are (up to ϵ^6)

$$\hat{L}u_0=0, \tag{20}$$

$$\hat{L}u_2=-2u_{0,\zeta\tau_3}+u_{0,2\tau_1}-3m\sqrt{\lambda}u_0^2, \tag{21}$$

$$\hat{L}u_4=u_{2,2\tau_1}-2u_{2,\zeta\tau_3}-2u_{0,\zeta\tau_5}+2u_{0,\tau_1\tau_3}-6m\sqrt{\lambda}u_0^2u_2-\lambda u_0^3, \tag{22}$$

$\hat{L}$ being the linear Klein-Gordon operator

$$\hat{L}(v)=2v_{\zeta\tau_1}+2m^2v. \tag{23}$$

For the solution u_0 of Eq. (20) we choose the form $u_0=B\exp 2\eta$ with $\eta=k_0\zeta-(4k_0)^{-1}m^2\tau_1+\omega_3\tau_3+\cdots$ and B a constant. All linear terms at the right hand side of the equations for $u_{2(n-1)}$ are secular. They can be eliminated choosing appropriately ω_{2n-1}, namely,

$$\omega_{2n-1}=-\frac{(\frac{1}{2})!}{n!(\frac{1}{2}-n)!}\frac{m^{2n}}{2^nk_0^{2n-1}}. \tag{24}$$

Next the solutions read

$$u_{2(n-1)}=B^n\left(\frac{\sqrt{\lambda}}{2m}\right)^{n-1}\exp 2n\eta. \tag{25}$$

With the values of ω_{2n-1} the series for η can again be summed as

$$\eta=kx-\sqrt{k^2+m^2/2}t,$$

as well as the perturbative series for u, but only if we choose $B=-(2m/\sqrt{\lambda})k_0^{-2}$. It leads for $\phi=u+m/\sqrt{\lambda}$ to

$$\phi=-\frac{m}{\sqrt{\lambda}}\tanh\{kx-\sqrt{k^2+m^2/2}t-\ln k\}. \tag{26}$$

To get this expression it is necessary to use the Fourier representation ($x<0$)

$$\sum_{n=0}^{\infty}(-1)^{n+1}\delta_n\exp(2nx)=\tanh x,$$

where δ_n are the Neumann's numbers ($\delta_0=1,\delta_n=2,\forall n=1,2,3,\ldots$).

The above solution ϕ is the antikink solution of ϕ^4 (with an initial shift ln k/k), which has never been obtained previously within another perturbation scheme.

The expression $\sqrt{k^2+m^2/2}$ can be interpreted as a nonlinear frequency ω_{nl}, which defines the nonlinear group velocity

$$v=\frac{\partial\omega_{nl}}{\partial k}=\frac{k}{\sqrt{k^2+m^2/2}}. \tag{27}$$

It is remarkable that the Lorentz invariance of Eq. (26) is precisely related to that particular velocity. Indeed

$$\phi=-\frac{m}{\sqrt{\lambda}}\tanh\left\{\frac{m}{\sqrt{2}}\left(\frac{xv}{\sqrt{1-v^2}}-\frac{t}{\sqrt{1-v^2}}\right)-\ln k\right\}. \tag{28}$$

Note that the case $\eta>0$ in the perturbative series would simply yield the solution $\phi(-\eta)$.

THE SG MODEL

Finally in the case of the sine-Gordon model (3), for $\phi=\epsilon(\phi_0+\epsilon^2\phi_2+\epsilon^4\phi_4+\cdots)$, with ϕ_{2n} functions of $\eta=k_0\zeta+(2k_0)^{-1}m^2\tau_1+\omega_3\tau_3+\cdots$, where $\zeta=\epsilon^{-1}(x-t)$, $\tau_1=\epsilon t,\tau_3=\epsilon^3t,\ldots$, we obtain (up to order ϵ^4)

$$\hat{L}(\phi_0)=0, \tag{29}$$

$$\hat{L}(\phi_2)=-2\phi_{0,\zeta\tau_3}+\phi_{0,2\tau_1}-\frac{\lambda}{3!}\phi_0^3, \tag{30}$$

$$\hat{L}(\phi_4)=-2\phi_{2,\zeta\tau_3}+\phi_{2,2\tau_1}-2\phi_{0,\zeta\tau_3}+2\phi_{0,\tau_1\tau_3}-\frac{\lambda}{3!}3\phi_0^2\phi_2+\frac{\lambda^2}{m^2}\frac{\phi_0^5}{5!}, \tag{31}$$

with $\hat{L}$ being the following operator

$$\hat{L}(v)=2v_{\zeta\tau_1}-m^2v. \tag{32}$$

For the solution of Eq. (32) we choose the expression $\phi_0=C\exp\eta$ with C a constant. As in the previous case all the linear terms at the right-hand side of the equations for $\phi_{2(n-1)}$ are secular. They can be eliminated by choosing $\omega_{2(n-1)}$ as

$$\omega_{2n-1}=(-1)^{n+1}\frac{(\frac{1}{2})!}{n!(\frac{1}{2}-n)!}\frac{m^{2n}}{k_0^{2n-1}}. \tag{33}$$

Hence the solutions $\phi_{2(n-1)}$ read

$$\phi_{2(n-1)}=-\left(\frac{-\lambda}{16}\right)^{n-1}\frac{C^{2n-1}\exp(2n-1)\eta}{m^{2(n-1)}(2n-1)}. \tag{34}$$

The series for ϕ sums for $C=4m/\sqrt{\lambda}k_0$ and yields

$$\phi=\frac{4m}{\sqrt{\lambda}}\sum_{n=0}^{\infty}(-1)^n\frac{\exp[(2n+1)(\eta-\ln k)]}{2n+1}=\frac{4m}{\sqrt{\lambda}}\arctan\{\exp(kx-\sqrt{k^2-m^2}t-\ln k)\}. \tag{35}$$

In this case the Lorentz invariant form of Eq. (35) appears as a function of the nonlinear phase velocity $v=\sqrt{1-m^2/k^2}$ as

$$\phi=\frac{4m}{\sqrt{\lambda}}\arctan\left\{\exp\left[\frac{m}{\sqrt{1-v^2}}(x-vt)-\ln k\right]\right\}. \tag{36}$$

CONCLUSION AND COMMENTS

We have applied a multiple-time version of the reductive perturbation method to study the solitary-wave and the kink-wave solutions of some nonlinear dispersive models. All these solutions have already been known before. The alternative way given here to obtain them shows that they represented a *short-wave asymptotic dynamics* $(t\to\infty)$.

(1) BBMP for long waves serves about the same purpose as KdV, whereas their behaviors in propagating short waves can be expected to be rather different. From a linear analysis, BBMP does not propagate short waves while KdV amplifies them [11]. Thus our result answers this old controversy on the relative relevance of KdV and BBMP [14]. Actually we proved that short waves do propagate nonlinearly in BBMP models, and build up soliton-like solutions as $t\to\infty$.

(2) The antikink (or kink) solution of the ϕ^4 model, which cannot be obtained as a perturbative solution in λ [12], appears as a perturbative solution in k in the short-wave limit.

(3) Equation (35) shows that the kink solution of SG is obtainable *only* from a short-wave dynamics, as the limit $k\to 0$ gives rise to an imaginary argument.

(4) An initial profile generically contains short-wave components that are usually neglected in favor of the long-wave components. This is the case whenever we realize numerical discretizations of the models. As we have shown, the short-wave components asymptotically build up soliton solutions. Therefore the common understanding of a soliton as originating from the long wave is to be questioned.

(5) It is worth noting finally that, in the long-wave approach, the nonlinear character of the solution is already present at first order, and we usually find the Boussinesq, KdV, MKdV, etc. equations. This is not the case with the short-wave approach where *all orders* are usually necessary to unveil the nonlinear character of the solution.

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